STEADY-STATE PERIODIC AND ROTATIONAL MOTIONS IN PERTURBED, SIGNIFICANTLY NONLINEAR AND ALMOST CONSERVATIVE SYSTEMS WITH ONE DEGREE OF FREEDOM, IN THE CASE OF AN ARBITRARY CONSTANT DEVIATION OF THE ARGUMENT

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The paper describes an investigation of the conditions of existence of the steady state periodic and rotational motions in a quasi-conservative system with one degree of freedom We formulate the sufficient conditions for the existence of a unique solution to a perturbed equation similar to the parent equation. Such conditions were obtained earlier for less general classes of equations.

1. Statement of the problem. We consider nonlinear systems described by equations of the type

$$x^{\cdot\cdot} + Q(x) = \varepsilon q(t, x, x^{\cdot}, x_\tau, x^{\cdot}_\tau; \varepsilon) \qquad (x_\tau = x(t-\tau),\ |\tau| < +\infty) \quad (1.1)$$

where $\varepsilon > 0$ is a small parameter, $t \in (-\infty, \infty)$ is an independent real variable and τ is a constant. We shall consider not only the basic, or perturbed equation, but also a degenerate form of (1.1), which is

$$x_0^{\cdot\cdot} + Q(x_0) = 0 \qquad (1.2)$$

and henceforth we shall assume that two-parameter families of solutions of (1.2) are given, which are either periodic $x_0 = \phi(\psi, \omega)$ or rotational

$$x_0 = \psi + \varphi(\psi, \omega) \qquad (1.3)$$

and where ϕ is a periodic function of t with the period $T_0 = 2\pi/\omega$, $\psi = \omega(t - t_0 + \theta)$, θ is an arbitrary phase constant, $\omega = \omega(E)$ is the frequency of the unperturbed periodic or rotational motion and E is the first integral of the unperturbed system [1 to 6].

It is well known, that the period of an unperturbed motion depends only on E and in the case of oscillations it is

$$T_0(E) = 2\int_{a_1}^{a_2} \frac{dx}{\sqrt{2[E - U(x)]}} \qquad \left(U(x) = \int Q(x)\,dx\right)$$

where $a_1(E)$ and $a_2(E)$ are simple real roots of Eq.

$$E - U(x) = 0 \quad (a_1 < a_2)$$

We shall assume the simplest case [2].

For the rotation, the expression for the period is somewhat simpler

$$T_0(E) = \int\limits_0^{2\pi} \frac{dx}{\sqrt{2[E - U(x)]}}$$

where 2π is the period of Q in x. It was proved in [5] that the solution of (1.2) will be rotational and of the type (1.3), provided that the function Q is periodic in x, that its mean value is zero and that $E > \max U$. When investigating the rotations in a perturbed system we should assume, that the function q is also periodic in x and x_τ, with the periods equal to 2π or $2\pi/n$ where n is an integer. In the oscillatory case the above assumptions need not be made. In both cases we assume that q is periodic with the period $\Pi =$ const and, that it is continuous in its argument t appearing in it explicitly. Assumptions concerning the smoothness of Q and q with respect to the remaining arguments, will be made later.

We shall also introduce the following assertion. The T-periodic solution of (1.1) will be of the resonant type m/n if the following equalities hold: $T = m\Pi = nT_0$. We should note that the latter relation defines the constant E.

We shall consider the resonant, steady-state periodic or rotational solutions of (1.1) for $t \in (-\infty, \infty)$ and below we investigate the conditions which are necessary for those motions to take place in the system. An analogous statement was employed by a large number of authors [7 and 8] (also see the bibliography in [7]) studying periodic solutions of quasi-linear systems with a deviating argument.

Nonlinear systems of the general type with time delay were studied in [9] for the particular case of an isolated generating periodic solution.

We should note that (1.1) can, be reduced by substitution to

$$dE / dt = \varepsilon f(t, E, E_\tau, \psi, \psi_\tau; \varepsilon), \quad d\psi / dt = \omega(E, E_\tau) + \varepsilon F(t, E, E_\tau, \psi, \psi_\tau; \varepsilon)$$

in which f and F are 2π-periodic in the rotating phases ψ and ψ_τ and Π-periodic in t. Autonomous system of the similar type with slowly varying parameters, was averaged in a similar context over the period of time $\sim 1/\varepsilon$ [10].

2. Construction of the perturbed solution. We shall use the method of consecutive approximations [2]. Assuming that Q has a second derivative in x and that q has first partial derivatives in x, $x^\cdot$, x_τ, $x_\tau^\cdot$ and ε in some vicinity of x_0 and x_0, $x_0^\cdot$, $x_{\tau 0}$, $x_{\tau 0}^\cdot$ and 0, respectively, which satisfy the Lipshits conditions and contain constants independent of t, we make the substitution $x = x_0 + \varepsilon y$ where y is an unknown periodic function This yields the following quasi-linear equation for y

$$y^{\cdot\cdot} + Q'(x_0)\, y = q(t, x_0, x_0^\cdot, x_{\tau,0}, x_{\tau,0}^\cdot; 0) + \varepsilon Y(t, y, y^\cdot, y_\tau, y_\tau^\cdot; \varepsilon) \qquad (2.1)$$

in which

$$Y(t, y, y^\cdot, y_\tau, y_\tau^\cdot; \varepsilon) = -\frac{1}{2} Q_0'' y^2 + \left(\frac{\partial q}{\partial x}\right)_0 y + \left(\frac{\partial q}{\partial x^\cdot}\right)_0 y^\cdot +$$
$$+ \left(\frac{\partial q}{\partial x_\tau}\right)_0 y_\tau + \left(\frac{\partial q}{\partial x_\tau^\cdot}\right)_0 y_\tau^\cdot + \left(\frac{\partial q}{\partial \varepsilon}\right)_0 + Y^*(t, y, y^\cdot, y_\tau, y_\tau^\cdot; \varepsilon)$$

and $Y^*(t, y, y^\cdot, y_\tau, y_\tau^\cdot; 0) \equiv 0$. We shall now construct a scheme of consecutive approximations in ε in order to obtain a periodic solution of (2.1). We shall obtain the zero approximation for y, assuming it to be a periodic solution of (2.1) at $\varepsilon = 0$.

$$y_0^{\cdot\cdot} + Q'(x_0)\, y_0 = q_0(t, x_0, x_0^\cdot, x_{\tau 0}, x_{\tau 0}^\cdot)$$

It is an ordinary linear inhomogeneous equation whose coefficients and the right-hand side are both periodic. Its integration presents no problems, since the basic method of solution of the corresponding homogeneous Eq.

$$y_{0,1} \equiv u = x_0^\cdot, \quad y_{0,2} = ut + v \qquad \left(v = \omega \frac{\partial x_0(\psi, \omega)}{\partial \omega}\right)$$

where u and v are periodic functions, is well known. Using the method of variation of the constants of integration [2 and 3] we obtain

$$y_0 \doteq D\left[u\int_{t_0}^{t}\left(\int_{t_0}^{t_1} q_0 u\, dt_2 - vq_0 - \beta_0\right)dt_1 + v\left(\int_{t_0}^{t} q_0 u\, dt_1 - \beta_0\right)\right] + \alpha_0 u \equiv$$
$$\equiv L\,[t, q_0] + \alpha_0 u \equiv y_0^* + \alpha_0 u \qquad (D = 1/\Delta(t) = 1/(u^2 + uv^{\cdot} - vu^{\cdot}))$$

where $\Delta(t)$ is a Wronskian which is constant by the Liouville's theorem, α_0 and β_0 are constants of integration and L is an operator linear in q_0.

It should be noted that the function y_0 will be T-periodic at any α_0, provided that the real constant θ satisfies Eq.

$$P(\theta) = \int_0^T q_0 u\, dt = 0 \tag{2.2}$$

and, that we put

$$\beta_0 = \frac{1}{T}\int_0^T\left(\int_{t_0}^{t} q_0 u\, dt_1 - vq_0\right)dt$$

Equation (2.2) defines the phase constant θ.

Next approximation for y is given by

$$y_1^{\cdot\cdot} + Q_0' y_1 = q_0 + \varepsilon Y\,(t, y_0, \dot{y}_0, y_{\tau,0}, y_{\tau,0}^{\cdot}; 0)$$

which, similarly to the previous one, has a solution of the form

$$y_1 = y_0^* + \varepsilon L\,[t, Y_0] + \alpha_1 u$$

Condition of periodicity of y_1, at any α_1, yields under some additional assumptions the constant α_0. Taking into account

$$-\frac{1}{2}\int_0^T Q''(x_0)\, y_0^2 u\, dt = \int_0^T q_0 y_0^{\cdot}\, dt$$

we can show by direct integration, that the equation defining α_0 is linear in α_0 and has the form

$$\alpha_0 \frac{\partial P}{\partial \theta^*} = -\int_0^T \left\{\left[\left(\frac{\partial q}{\partial x}\right)_0 y_0^* + \left(\frac{\partial q}{\partial x^{\cdot}}\right)_0 y_0^{\cdot *} + \left(\frac{\partial q}{\partial x_\tau}\right)_0 y_{\tau,0}^* + \right.\right.$$
$$\left.\left. + \left(\frac{\partial q}{\partial x_\tau^{\cdot}}\right)_0 y_{\tau,0}^{\cdot *} + \left(\frac{\partial q}{\partial \varepsilon}\right)_0\right] u + q_0 y_0^{\cdot *}\right\} dt$$

which yields α_0 by elementary operations, provided of course that θ^* is a simple, real root of (2.2).

To obtain further approximations for the periodic function y we shall use, in accordance with our method, the following Eqs. (where $i \geqslant 2$)

$$y_i^{\cdot\cdot} + Q_0' y_i = q_0 + \varepsilon Y\,(t, y_{i-1}, y_{i-1}^{\cdot}, y_{\tau,i-1}, y_{\tau,i-1}^{\cdot}; \varepsilon) \tag{2.3}$$

whose solution is

$$y_i = y_0^* + \varepsilon L\,[t, Y_{i-1}] + \alpha_i u \tag{2.4}$$

Condition of periodicity of y_i yields, as before, the unknown constant α_{i-1} or in other words, the $(i-1)$-th approximation in ε for all $t \in (-\infty, \infty)$ under a single condition that these successive approximations converge uniformly and belong to the domain of definition of the function Y. We should note that the equations defining α_k $(k \geqslant 1)$ will be nonlinear and of the form

$$\alpha_k \frac{\partial P}{\partial \theta^*} + \int_0^T \left\{ \left[\left(\frac{\partial q}{\partial x}\right)_0 y_k^* + \left(\frac{\partial q}{\partial x^\cdot}\right)_0 y_k^{\cdot *} + \left(\frac{\partial q}{\partial x_\tau}\right)_0 y_{\tau, k}^* + \left(\frac{\partial q}{\partial x_\tau^\cdot}\right)_0 y_{\tau, k}^{\cdot *} + \right.\right.$$

$$\left.\left. + \left(\frac{\partial q}{\partial \varepsilon}\right)_0 + Y_k^* \right] u + q_0 y_k^{\cdot *} + \varepsilon (y_k^{\cdot *} + \alpha_k u^\cdot) Y_{k-1} \right\} dt = 0 \qquad (2.5)$$

Since (2.5) satisfies all the requirements of the theory of existence of the implicit function $\alpha_k(\varepsilon)$, we may be justified in saying, that, for a sufficiently small $|\varepsilon|$ there exists a unique solution $\alpha_k = \alpha_k(\varepsilon)$ of Eq. (2.5) and that $\alpha_k(0) = \alpha_0$. This solution can be constructed using the method of successive approximations according to the scheme

$$\alpha_k^{(j)} = -\left(\frac{\partial P}{\partial \theta^*}\right)^{-1} \int_0^T \left\{ \left[\left(\frac{\partial q}{\partial x}\right)_0 y_k^* + \left(\frac{\partial q}{\partial x^\cdot}\right)_0 y_k^{\cdot *} + \left(\frac{\partial q}{\partial x_\tau}\right)_0 y_{\tau, k}^* + \left(\frac{\partial q}{\partial x_\tau^\cdot}\right)_0 y_{\tau, k}^{\cdot *} + \right.\right.$$

$$\left.\left. + \left(\frac{\partial q}{\partial \varepsilon}\right)_0 + Y_k^{(j-1)*} \right] u + q_0 y_k^{\cdot *} + \varepsilon (y_k^{\cdot *} + \alpha_k^{(j-1)} u^\cdot) Y_{k-1} \right\} dt$$

$$(j = 1, 2, \ldots; \ \alpha_k^{(0)} = \alpha_0)$$

It therefore follows that the proposed scheme allows us to obtain, uniquely, any degree of the formal approximation in ε to the periodic solution of (2.1) for all $t \in (-\infty, \infty)$. This can easily be proved using the method of induction. Next we shall prove the validity of the scheme (2.3).

3. Proof of the validity of the scheme of successive approximations. We shall use the method developed in [2 and 11].

First we shall discuss the basic properties of the operator L. L is a linear operator satisfying, by virtue of periodicity, the condition

$$\max |L[t, F]| < A \cdot B \quad (B > 0) \qquad (3.1)$$

where $A = \max|F|$, while the constant B is bounded and does not depend on the choice of F; moreover, the properties of smoothness of the function in terms of the arguments entering F, are not affected.

We shall further introduce the notation

$$\sigma_k = \sigma_k(t, \alpha, \varepsilon) = y_0^* + \varepsilon L[t, Y_{k-1}] + \alpha u$$

$$R_k = R_k(\alpha, \varepsilon) = \int_0^T Y(t, \sigma_k, \sigma_k^\cdot, \sigma_{\tau, k}, \sigma_{\tau, k}^\cdot; \varepsilon) u dt$$

with the help of which we can write the equation defining α_k as

$$R_k(\alpha_k, \varepsilon) = 0 \qquad (3.2)$$

We shall first show that when ε is sufficiently small, then the functions y_k, $y_k^\cdot$, $y_{\tau,k}$ and $y_{\tau,k}^\cdot$ belong, for all $t \in (-\infty, \infty)$, to some bounded region G, provided that α_0 is such that y_0, $y_0^\cdot$, $y_{\tau,0}$ and $y_{\tau,0}^\cdot$ belong to the same region. To prove it we shall assume that it is valid for all k up to $(k-1)$ inclusive, and then we shall show that when ε is sufficiently small and independent of k, then the boundedness property is also valid for k. From the fundamental property (3.1) of the operator L we have, that $|L[t, Y_{k-1}]| < A \cdot B$ where $A = \max|Y|$ over the whole domain of definition of Y. Further

$$\left.\frac{\partial R_k}{\partial \alpha}\right|_{\varepsilon=0, \alpha=\alpha_0} = \frac{\partial P}{\partial \theta^*} \neq 0$$

But then we can easily deduce from Expression

$$\frac{\partial R_k}{\partial \alpha} = \int_0^T \left(\frac{\partial Y}{\partial \sigma_k} u + \frac{\partial Y}{\partial \sigma_k^\cdot} u^\cdot + \frac{\partial Y}{\partial \sigma_{\tau, k}} u_\tau + \frac{\partial Y}{\partial \sigma_{\tau, k}^\cdot} u_\tau^\cdot \right) u \, dt$$

that two positive numbers μ and η_1 independent of k exist, such that when

$$|\alpha - \alpha_0| < \mu \tag{3.3}$$

and $\varepsilon < \eta_1$ then the inequality

$$|\partial R_k / \partial\alpha| > \gamma \tag{3.4}$$

where $\gamma > 0$ is independent of k, holds. We shall assume here that μ and η_1 are so small, that $(\sigma_k, \sigma_k^{\cdot}, \sigma_{\tau,k,k}, \sigma_{\tau,k}^{\cdot}) \in G$.

We shall now assume that when $\varepsilon < \eta_1$, then the roots $\alpha_k(\varepsilon)$ of (3.2) lie within the region (3.3), and we shall show that the magnitude η_1 can indeed be chosen small enough to ensure that

$$|\alpha_k(\varepsilon) - \alpha_0| < \mu \tag{3.5}$$

holds.

Let now $C = \max|\partial R_k/\partial\varepsilon|$, assuming that C is independent of k over the whole domain of existence of this derivative, and let us put $\eta_1 < \gamma\mu C$. Then the inequality (3.5) will certainly hold. Indeed, since $\alpha_k(0) = \alpha_0$, the inequality (3.5) by virtue of continuous dependence will hold at sufficiently small ε. Let us now assume the opposite, i.e. that at some $\varepsilon = \varepsilon^*$ (3.5) becomes an equality. We shall show that this is possible when $\varepsilon^* > \eta_1$, assuming initially the opposite, i.e. $\varepsilon^* \leqslant \eta_1$. We can then write

$$|\alpha_k(\varepsilon^*) - \alpha_0| = |\alpha_k(\varepsilon^*) - \alpha_k(0)| =$$
$$= \varepsilon^* \left|\frac{\partial\alpha_k(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varkappa_1\varepsilon^*} = \varepsilon^* \left|\frac{\partial R_k}{\partial\varepsilon}\left(\frac{\partial R_k}{\partial\alpha}\right)^{-1}\right|_{\alpha=\alpha_k(\varkappa_1\varepsilon^*),\ \varepsilon=\varkappa_1\varepsilon^*}$$

where $\varkappa_1$ is a proper positive fraction. Since $\alpha_k(\varkappa_1\varepsilon^*)$ lies within the region (3.5), the inequality (3.4) yields

$$|\alpha_k(\varepsilon^*) - \alpha_0| < \varepsilon^* C/\gamma \leqslant \eta_1 C/\gamma < \mu$$

which contradicts the assumption that (3.5) became an equality. Thus we have shown that, when the condition $\varepsilon < \eta_1$ holds and η_1 is chosen as required, all approximations belong to G. Now we shall prove that the consecutive approximations (2.4) converge uniformly. Let us introduce the following differences

$$|\alpha_k(\varepsilon) - \alpha_{k-1}(\varepsilon)| < b_k; \quad |L[t, Y_{k-1}] - L[t, Y_{k-1}]| < a_k$$
$$L^{\cdot}[t, Y_{k-1} - Y_{k-2}] | < v_k; \quad |L_\tau[t, Y_{k-1} - Y_{k-2}]| < a_k^\tau; \quad |L_\tau^{\cdot}[t, Y^-{}_{k-1} Y_{k-2}]| < v_k^\tau$$

where b_k, a_k, v_k, a_k^τ and v_k^τ are some positive constants with upper bounds independent of k. Let c_k be the largest of a_k, v_k, a_k^τ and v_k^τ. Since the function Y satisfies, in G, the Lipshits conditions, we have

$$|Y_k - Y_{k-1}| < 4\Omega(b_k\lambda + \varepsilon c_k)$$

where λ denotes the maximum value of the following periodic functions $|u|$, $|u^{\cdot}|$, $|u_\tau|$, and $|u_\tau^{\cdot}|$, while Ω is the Lipshits constant. Taking into account the above inequality we find

$$|L[t, Y_k - Y_{k-1}]| < c_{k+1}, \quad |L^{\cdot}[tY_k - Y_{k-1}]| < c_{k+1}, \quad |L_\tau[t, Y_k - Y_{k-1}]| < c_{k+1}$$
$$|L_\tau^{\cdot}[t, Y_k - Y_{k-1}]| < c_{k+1} \qquad (c_{k+1} = 4\Omega B_*(\lambda b_k + \varepsilon c_k))$$

Let us now obtain an estimate for the difference $\alpha_{k+1}(\varepsilon) - \alpha_k(\varepsilon)$. We begin by considering the following auxilliary Eq.:

$$\Phi_k(\beta_k, \varepsilon, \delta) = 0 \tag{3.6}$$

where

$$\Phi_k(\beta, \varepsilon, \delta) = \int_0^T (t, \xi_k, \xi_k^{\cdot}, \xi_{\tau,k}, \xi_{\tau,k}^{\cdot}; \varepsilon)\, u\, dt$$

$$\xi_k = y_0^* + \beta u + \varepsilon L[t, Y_{k-1}] + \delta L[t, Y_k - Y_{k-1}], \quad \delta \in [0, \varepsilon]$$

We similarly define the functions $\xi_k^{\cdot}$, $\xi_{T,k}$ and $\xi_{T,k}^{\cdot}$. Obviously, $\eta_2(\varepsilon < \eta_2)$ can be chosen sufficiently small to ensure that $(\xi_k, \xi_k^{\cdot}, \xi_{T,k}, \xi_{T,k}^{\cdot}) \in G$, if

$$|\beta - \alpha_0| < \nu \tag{3.7}$$

where ν is positive and sufficiently small. We can then assume that the function Φ_k is fully defined. We shall now show that when η_2 is sufficiently small, then the inequality $|\beta_k - \alpha_0| < \nu$ where $\beta_k(\varepsilon, \delta)$ is a root of (3.6), holds for any k. Indeed we have

$$|\beta_k(\varepsilon, \delta) - \alpha_0| < \mu + |\beta_k(\varepsilon, \delta) - \beta_k(\varepsilon, 0)|$$

We can obtain $\mu < \nu$ by choosing a sufficiently small η_1. We have then $\nu - \mu = \Delta > 0$. We shall further show that $|\beta_k(\varepsilon, \delta) - \beta_k(\varepsilon, 0)| < \Delta$ provided that η_2 is sufficiently small. Putting

$$\eta_2 < \omega\Delta / M \qquad (M = \max|\partial\Phi_k/\partial\delta|;\ 0 < \omega < \partial\Phi_k / \partial\beta)$$

where none of the magnitudes depend on k we obtain, in analogy to the previous case, the required assertion. Now we can estimate the difference $\alpha_{k+1}(\varepsilon) - \alpha_k(\varepsilon)$, noting that $\alpha_{k+1}(\varepsilon) = \beta_k(\varepsilon, \varepsilon)$ and that $\alpha_k(\varepsilon) = \beta_k(\varepsilon, 0)$. We obtain

$$|\alpha_{k+1}(\varepsilon) - \alpha_k(\varepsilon)| = |\beta_k(\varepsilon, \varepsilon) - \beta_k(\varepsilon, 0)| = \varepsilon\left|\frac{\partial\beta_k(\varepsilon, \delta)}{\partial\delta}\right|_{\delta=\varkappa_2\varepsilon} = \varepsilon\left|\frac{\partial\Phi_k}{\partial\delta}\left(\frac{\partial\Phi_k}{\partial\beta}\right)^{-1}\right|$$

in which $\beta = \beta_k(\varepsilon, \varkappa_2\varepsilon)$, $\delta = \varkappa_2\varepsilon$ and $0 < \varkappa_2 < 1$. Let us now estimate $\partial\Phi_k/\partial\delta$. Differentiating Φ_k we find, that in the region (3.7) and when $\varepsilon < \eta_2$, we have

$$|\partial\Phi_k / \partial\delta| < 4T\lambda H c_{k+1} \equiv W c_{k+1}$$

where H is the largest of the upper bounds of $\partial Y/\partial y$, $\partial Y/\partial y^{\cdot}$, $\partial Y/\partial y_T$ and $\partial Y/\partial y_T^{\cdot}$ in their domain of existence.

Collecting the estimates we now obtain

$$|\alpha_{k+1}(\varepsilon) - \alpha_k(\varepsilon)| < b_{k+1} \qquad (b_{k+1} = \varepsilon W c_{k+1} / \omega)$$

from which it follows that the ratio $b_{k+1}/C_{k+1} = \varepsilon W/\omega$ and is independent of k. Consequently we can infer that the ratio b_k / c_k is also independent of k. But then the ratios b_{k+1}/b_k and c_{k+1}/c_k are also independent of k, since

$$\frac{b_{k+1}}{b_k} = \varepsilon\,\frac{W}{\omega}\,\frac{c_{k+1}}{b_k} = 4\varepsilon\Omega B\,\frac{W}{\omega}\left(\lambda + \frac{c_k}{b_k}\varepsilon\right), \qquad \frac{c_{k+1}}{c_k} = 4\Omega B\left(\lambda\,\frac{b_k}{c_k} + \varepsilon\right)$$

As b_k/c_k is proportional to ε, we find that when ε is sufficiently small, the ratios c_{k+1}/c_k and b_{k+1}/b_k will be less than unity, which proves that $y_k(t, \varepsilon)$ converges absolutely and uniformly.

Finally we shall show that the limit of this sequence is a solution of (2.1). Since L and Y are smooth, we have

$$\varepsilon L[t, Y] = \varepsilon \lim_{k\to\infty} L[t, Y_{k-1}] = \lim_{k\to\infty}(y_k - \alpha_k u - y_0^*) = y(t, \varepsilon) - \alpha(\varepsilon) u - y_0^*$$

Differentiating it we find, by the uniform convergence, that $y(t, \varepsilon)$ is a periodic solution of (2.1).

Thus we have constructed a unique, resonant solution of (1.1) of the form m/n, with an arbitrary constant deviation of its argument, for the rotational and oscillatory cases, in the form

$$x = x(t, \varepsilon) = x_0(\psi, \omega) + \varepsilon y(t, \varepsilon) \tag{3.8}$$

where y is a T-periodic function, and this proves the following theorem.

T h e o r e m 3.1. When the values of the parameter ε are sufficiently small, then the perturbed Eq. (1.1) allows, in the region of definition and smoothness of the function Q, a unique, m/n, resonant, oscillatory or rotational solution stationary for all $t \in (-\infty, \infty)$, which becomes the generating solution $x_0(\psi, \omega)$ when $\varepsilon = 0$ and which has the form (3.8) provided that:

1) functions Q and q satisfy the periodicity and smoothness conditions listed in Sections 1 and 2;

2) equation (2.2) has a real root θ^* and

3) the relation $\omega'(E^*)\partial P/\partial\theta^* \neq 0$ holds.

Note 3.1. The uniqueness of the solution is understood in the sense, that for each real simple root θ^* corresponding to some fixed τ, m, n and ε, there exists one solution of the form (3.8). It is easily seen that a given segment of length T, always contains an even number of such roots, i.e. 0, 2, 4,... .

Note 3.2. Let θ^* be an r-tuple ($r < \infty$) real root, i.e.

$$\frac{\partial P}{\partial\theta^*} = \frac{\partial^2 P}{\partial\theta^{*2}} = \ldots = \frac{\partial^{r-1} P}{\partial\theta^{*r-1}} = 0, \qquad \frac{\partial^r P}{\partial\theta^{*r}} \neq 0$$

(we naturally assume that Q and q can be differentiated sufficient number of times). In this case the solution may be no longer unique in the above sense and we then approximate the exact solution, as a rule, in fractional powers of ε. Obviously we can obtain the result $\partial P/\partial\theta^* \neq 0$ by changing the constant τ and some other parameters, but this case was shown in [2 and 3] to be critical and seldom met in practice. It should be noted that the critical cases for an analytic, autonomous, quasi-linear equation without a deviating argument were investigated in [12].

Note 3.3. Another particular case which is more common occurs, when the equation (2.2) is satisfied identically, i.e. independently of θ for the given choice of τ and m, n. In this case we speak of the higher degree motions. Such oscillatory and rotational motions in the systems described by ordinary equations of the type (1.1), were studied in [3 and 13].

Note 3.4. The case $\omega'(E^*) = 0$ requires a separate investigation.

4. Example. To illustrate the method, we shall consider the following real system described by a 'pendulum' equation

$$x'' + a^2 \sin x = \varepsilon\,[N \sin \nu t + b x'(t-\tau) - \beta x' - \alpha \operatorname{sgn} x']\ (a^2, N, b, \beta, \alpha = \text{const} > 0)$$

whose generating solution has, in the rotational case (if $E > 2a^2$) the form

$$x_0 = 2\, \mathrm{am}\,[\sqrt{E/2}\,(t+\theta),\ a\sqrt{2/E}] = \psi + 4\sum_{p=1}^{\infty} \frac{1}{p}\,\frac{q^p}{1+q^{2p}} \sin p\psi$$

$$\left(\psi = \omega(E)(t+\theta), \qquad T_0(E) = 2\sqrt{2/E}\,K(a\sqrt{2/E}), \qquad q = \exp\frac{-\pi K'}{K}\right)$$

Here am is an elliptic amplitude, K denotes a complex elliptic integral of the first kind and K' denotes its derivative [14]. We shall for simplicity limit ourselves to the principal resonance $\omega(E) = \nu$. After a cumbersome integration we obtain the following condition for the phase equilibrium

$$P(\theta) = \frac{\pi}{\nu}\Big[-\frac{Nq}{1+q^2}\sin\nu\theta + 16\nu\sum_{p=1}^{\infty}\frac{q^{2p}}{(1+q^{2p})^2}(b\cos p\nu\tau - \beta) +$$

$$+ 2\nu(b-\beta) - 2\alpha\Big] \equiv \frac{\pi}{\nu}\Big(-\frac{Nq}{1+q^2}\sin\nu\theta + \gamma\Big) = 0$$

which has simple real roots on the segment $[0, 2\pi/\nu]$: $\theta_1 = (1/\nu)\arcsin\delta$ and $\theta_2 = \pi/\nu - \theta_1$ ($\delta = \gamma/N)\,(q + 1/q)$) provided that $\delta < 1$. When $\gamma = Nq/(1+q^2)$, we easily find that $\partial P/\partial\theta^* = 0$. If on the other hand $\delta < 1$, then the basic resonant rotation cannot take place near $\varepsilon = 0$. Thus, if $\gamma < Nq/(1+q^2)$, then by our theorem there exists a basic resonant solution of the perturbed equation. Further deductions can be made without any fundamental difficulties, using the formulas of Section 2.

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